

A generalized photon propagator

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A covariant gauge independent derivation of the generalized dispersion relation of electromagnetic waves in a medium with local and linear constitutive law is presented. A generalized photon propagator is derived. For Maxwell constitutive tensor, the standard light cone structure and the standard Feynman propagator are reinstated.

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I. INTRODUCTION

From a formal point of view [1], [2], the Maxwell electrodynamics theory can be represented by a system of two independent equations

$$\epsilon^{ijkl} F_{jk,l} = 0, \quad H^{ij}{}_{,j} = J^i, \quad (1.1)$$

where two independent antisymmetric tensors, the field strength tensor F_{ij} and the excitation tensor density H^{ij} are involved. The electromagnetic current vector density is denoted by J^i . Here the commas stand for ordinary derivatives, the indices range from 0 to 3, the Levi-Civita permutation tensor is normalized by $\epsilon^{0123} = 1$.

For most applications it is enough to assume a local, linear, homogeneous constitutive relation between the fields F_{ij} and H^{ij} ,

$$H^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}. \quad (1.2)$$

By the definition, the constitutive tensor χ^{ijkl} has to respect the symmetries of the fields F_{ij} and H^{ij} ,

$$\chi^{ijkl} = \chi^{[ij]kl} = \chi^{ij[kl]}. \quad (1.3)$$

Hence it has, in general, 36 independent components.

The standard Maxwell electrodynamics in vacuum is reinstated in this formalism by a special choice of the Maxwell-Lorentz constitutive tensor

$$^{(\text{Max})} \chi^{ijkl} = \lambda_0 \sqrt{-g} (g^{ik} g^{jl} - g^{il} g^{jk}). \quad (1.4)$$

Here g^{ij} is the Lorentz metric components, while λ_0 is a constant with the dimension of an admittance.

In this paper, we study the formal scheme (1.1), (1.2) with a general constitutive tensor χ^{ijkl} . The physical spacetime is considered as a bare manifold without metrics or connection. All the information about the geometry of this space is encoded in the constitutive tensor. In other words we are dealing with a premetric electrodynamics.

Such a construction is applicable for description of a rather wide range of physics effects. As a classical field theory, the premetric electrodynamics involves the standard Maxwell electrodynamics in vacuum and even provide a possibility to describe the additional degrees of

freedom (axion, dilaton and skewon) as the premetric partners of photon [3], [4]. Moreover, since the metric is a secondary quantity in this scheme, its form [5], [6], [7], [8], and the signature [9] are derived from the properties of the constitutive tensor. The nonminimal coupling of the electromagnetic field to the torsion yields the birefringence of vacuum [10], [11]. This effect finds its natural description in the premetric scheme, [12], [13].

Another interesting area of application is the models with violation of Lorentz invariance. In particular the Carroll-Field-Jackiw modification of the Maxwell electrodynamics [14], see also [15], is embedded in the premetric scheme. The wave propagation in this model requires, however, to go beyond the geometrical optics approximation [16]. This problem will be considered in a contributed publication.

The mathematical methods similar to used here was shown to be useful in ray optics applications to GR [17] and in quantum plasmadynamics [18].

In the present letter, we give a covariant gauge independent derivation of the generalized dispersion relation for the premetric electrodynamics. Moreover, we derive a generalized Green function in the momentum representation – a generalized photon propagator.

II. DISPERSION RELATION

To study the wave propagation in the premetric electrodynamics model, we solve the first equation of (1.1) in term of potentials $F_{ij} = (1/2)(A_{i,j} - A_{j,i})$. Substituting it into (1.2) and (1.1) and the current J^i to be equal to zero we derive

$$\chi^{ijkl} A_{k,lj} = 0. \quad (2.1)$$

To study the wave-type solutions of this equation we consider an ansatz

$$A_{ij}(x) = a_i e^{i\varphi}, \quad (2.2)$$

where $\varphi = \varphi(x^i)$ while a_i is a constant covector. Such solutions always exist on sufficiently small neighborhoods even on bare manifold [17]. Denote the wave covector as $q_i = \varphi_{,i}$.

In the geometrical optics approximation, the changes of the media parameters are neglected relative to the changes of the wave characteristics. Consequently we come to an algebraic system

$$M^{ik}a_k = 0, \quad \text{where} \quad M^{ik} = \frac{1}{2}\chi^{ijkl}q_lq_j. \quad (2.3)$$

Due to the symmetries of the constitutive tensor (1.3) the matrix of the system satisfies

$$M^{ik}q_k = 0, \quad M^{ik}q_i = 0. \quad (2.4)$$

The first relation of (2.4) means the *gauge freedom* of the vector potential while the second relation is interpreted as a *charge conservation condition*. Due to (2.4), the rows (and the columns) of the matrix M^{ij} are linearly dependent, so its determinant is equal to zero. Moreover, the gauge relation (2.4) can be interpreted as a fact that

$$a_k = Cq_k \quad (2.5)$$

is a formal solution of (2.3). This solution does not give a contribution to the electromagnetic field strength so it is unphysical.

An additional physically meaningful solution has to be linear independent on (2.5). A linear system has two or more linear independent solutions if and only if its rank is two (or less). Consequently, a generalized electrodynamics system has a physically meaningful solution if

$$A_{ij} = 0. \quad (2.6)$$

Here we involved the adjoint matrix A_{ij} – a matrix constructed from the cofactors of M^{ij} . The components of the adjoint matrix are expressed by the derivatives of the determinant relative to the entries of the matrix

$$A_{ij} = \frac{\partial \det(M)}{\partial M^{ij}} = \frac{1}{3!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3}. \quad (2.7)$$

Since the adjoint matrix has, in general, 16 independent components it seems that we have to require 16 independent conditions. The following algebraic fact shows that the situation is rather simpler.

Proposition: *If a square $n \times n$ matrix M^{ij} satisfies the relations*

$$M^{ij}q_i = 0, \quad M^{ij}q_j = 0 \quad (2.8)$$

for some nonzero vector q_i , its adjoint matrix A_{ij} is represented by

$$A_{ij} = \lambda(q)q_iq_j. \quad (2.9)$$

For a formal proof of this fact, see [20]. Consequently, instead of (2.6), we have only one condition

$$\lambda(q) = 0. \quad (2.10)$$

This condition is necessary to have physically meaningful solutions of the generalized wave equation, so it is a generalized dispersion relation.

The problem now is to derive from (2.9) the explicit expression for the function $\lambda(q)$. It is provided [20] by using the fact that the functions involved in (2.9) are homogeneous polynomials. In fact, A_{ij} is of the sixth order in the wave covector q^i , while $\lambda(q)$ is of the fourth order. Applying twice the derivatives with respect to the components of the covector q^i and using Euler's rule for the homogeneous functions, we obtain

$$\lambda(q) = \frac{1}{72} \frac{\partial^2 A_{ij}}{\partial q_i \partial q_j}. \quad (2.11)$$

In term of the matrix M^{ij} , the function $\lambda(q)$ is rewritten as

$$\lambda(q) = \frac{1}{144} \epsilon_{ii_1i_2i_3} \epsilon_{jj_1j_2j_3} \left(\frac{\partial^2 M^{i_1j_1}}{\partial q_i \partial q_j} M^{i_2j_2} + 2 \frac{\partial M^{i_1j_1}}{\partial q_i} \frac{\partial M^{i_2j_2}}{\partial q_j} \right) M^{i_3j_3}. \quad (2.12)$$

This expression may be useful for actual calculations of the dispersion relation for different media.

In order to have an explicit expression of the function λ in term of the constitutive tensor we have to calculate the corresponding derivatives. The resulting dispersion relation is

$$\epsilon_{ii_1i_2i_3} \epsilon_{jj_1j_2j_3} \left(\chi^{i_1(ij)j_1} \chi^{i_2abj_2} + 4 \chi^{i_1(ia)j_1} \chi^{i_2(jb)j_2} \right) \chi^{i_3cdj_3} q_a q_b q_c q_d = 0. \quad (2.13)$$

This equation is completely equivalent to the recently proposed [2] covariant dispersion relation

$$\epsilon_{ii_1i_2i_3} \epsilon_{jj_1j_2j_3} \chi^{ii_1ja} \chi^{bi_2j_1c} \chi^{di_3j_2j_3} q_a q_b q_c q_d = 0. \quad (2.14)$$

Indeed, in the special coordinate basis with $q_i = (q, 0, 0, 0)$, both equations yield the same non-covariant expression. Also the direct proof of the equivalence of two forms is acceptable [19].

The function $\lambda(q)$ is a fourth order polynomial. When it is separated to a product of two non-positive defined quadratic factors birefringence effect emerges. This effect is well known from the classical optics. However, in the premetric approach two light cones explicitly represent violation of Lorentz invariance. The non-birefringence condition can be given [7] in a rather simple covariant form: For an arbitrary covector q ,

$$\lambda(q) \geq 0 \quad (2.15)$$

has to be satisfied. For a component-wise representation of this condition, see also [6].

Observe an important special case. When the skewon part absents, the constitutive tensor respects the symmetries

$$\chi^{ijkl} = \chi^{klij}. \quad (2.16)$$

In this case, two terms in (2.13) are proportional one to another. Thus two additional expressions for the restricted dispersion relation emerge

$$\epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \chi^{i_1(ij)j_1} \chi^{i_2 ab j_2} \chi^{i_3 cd j_3} q_a q_b q_c q_d = 0, \quad (2.17)$$

and

$$\epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \chi^{i_1(ia)j_1} \chi^{i_2(jb)j_2} \chi^{i_3 cd j_3} q_a q_b q_c q_d = 0. \quad (2.18)$$

For the Maxwell constitutive tensor (1.4), the matrix M^{ij} takes the form

$$^{(\text{Max})} M^{ij} = \lambda_0 \sqrt{-g} (g^{ij} q^2 - q^i q^j). \quad (2.19)$$

The corresponding adjoint matrix is

$$^{(\text{Max})} A_{ij} = -(\lambda_0 \sqrt{-g})^3 q^4 q_i q_j. \quad (2.20)$$

Consequently, in this special case, the dispersion relation takes its regular form $q^2 = 0$.

III. PHOTON PROPAGATOR

Let us return to the full inhomogeneous Maxwell equation with a non-zero current. In the "momentum" representation, it takes the form

$$M^{ik} a_k = j^i. \quad (3.1)$$

Observe that the charge conservation law is expressed now as

$$j^i q_i = 0. \quad (3.2)$$

It is useful to have a formal solution of the equation (3.1) for an arbitrary given current j_k . Such a solution is usually given by the Green function or photon propagator, $D_{ij}(q)$. This tensor is defined in such a way that the covector

$$a_k = -D_{ki} j^i \quad (3.3)$$

is a formal solution of (3.1). Note that, due to the gauge invariant and charge conservation, the propagator, $D_{ij}(q)$, is defined only up to addition of terms proportional to the wave covector q_i ,

$$D_{ij} \rightarrow D_{ij} + \phi_i q_j + \psi_j q_i. \quad (3.4)$$

Here the components of the covectors ϕ_i and ψ_i are arbitrary functions of the wave covector. In the standard electrodynamics, an expression for this quantity is known as the Feynman propagator

$$D_{ij} = -\frac{g_{ij}}{\lambda_0 q^2 \sqrt{-g}}, \quad (3.5)$$

Note that it is expressed by a symmetric matrix. Thus also the covectors ϕ_i and ψ_i are usually taken to be equal

one to another. In our general setting, D_{ij} can be asymmetric. Consequently it is useful to preserve two arbitrary covectors in (3.4).

Substituting (3.3) into (3.1) we get

$$(M^{ik} D_{km} - \delta_m^i) j^m = 0, \quad (3.6)$$

Note that the matrix M^{ik} is singular, so the propagator cannot be taken to be proportional to the inverse of M^{ik} . In the standard case, it is not a problem. Indeed, with the matrix M^{ij} given by (2.19) and with the Feynman propagator (3.5) we have

$$M^{ik} D_{km} = -\delta_m^i + \lambda_0 \frac{q^i q_m}{q^2} \sqrt{-g}. \quad (3.7)$$

When this expression is substituted in (3.6) the second term disappears due to the charge conservation equation (3.2) and the equation is satisfied.

Our task is to derive an expression for the photon propagator in a general case when the metric tensor is not acceptable. To deal with the singular matrix M^{ij} , we consider the tensor density

$$B_{ijkl} = \frac{\partial A_{ij}}{\partial M^{kl}} = \frac{\partial \det(M)}{\partial M^{kl} \partial M^{ij}}. \quad (3.8)$$

This is the, so called, second adjoint (or the second adjugate compound) of the matrix M^{ij} . It is obtained by removing two arbitrary rows and two arbitrary columns from the original matrix. Observe that due to its definition the second adjoint tensor respects the symmetry

$$B_{ijkl} = B_{klij} \quad (3.9)$$

It is expressed by the components of the matrix M^{ij} as

$$B_{ijkl} = \frac{1}{2} \epsilon_{ik i_1 i_2} \epsilon_{jl j_1 j_2} M^{i_1 j_1} M^{i_2 j_2}. \quad (3.10)$$

From this expression, we read off the additional symmetries

$$B_{ijkl} = -B_{kjil} = -B_{ilkj}. \quad (3.11)$$

Let us derive an identity involving the second adjoint tensor. The derivative of the generalized Laplace expansion $A_{ij} M^{ik} = 0$ relative to the entries of the matrix M^{rs} yields

$$B_{ijrs} M^{ik} = -A_{rj} \delta_s^k. \quad (3.12)$$

We multiply now both sides of the equation (3.1) by the tensor B_{ijrs} to get

$$B_{ijrs} M^{ik} a_k = B_{ijrs} j^i. \quad (3.13)$$

Using (3.12) we rewrite it as

$$A_{rj} a_s = -B_{ijrs} j^i. \quad (3.14)$$

Substituting (2.9) we get

$$\lambda q_m q_n a_k = -B_{imnk} j^i. \quad (3.15)$$

We are coming once more to the same problem: How to "divide" both sides of this equation by the covector q_i in a covariant manner? Observe that λ and B_{imnk} are homogeneous polynomials in q of the order 4. Assuming j^i to be independent on q , we see that a_k is a homogeneous polynomial in q of the order -2 . Note that this is in a correspondence with the classical expressions (3.5).

Applying twice the partial derivatives with respect to the components of the wave covector and using Euler's rule for the homogeneous functions we come to

$$a_k = -\frac{1}{6} \frac{\partial^2}{\partial q_m \partial q_n} \left(\frac{B_{imnk}}{\lambda} \right) j^i. \quad (3.16)$$

Consequently we derived an expression for the generalized photon propagator

$$D_{ij} = \frac{1}{6} \frac{\partial^2}{\partial q_m \partial q_n} \left(\frac{B_{mijn}}{\lambda} \right). \quad (3.17)$$

Using the homogeneity of the polynomials involved here we get certain equivalent expressions

$$D_{ij} = \frac{1}{42\lambda} \frac{\partial^2 B_{mijn}}{\partial q_m \partial q_n} = \frac{1}{42\lambda} \frac{\partial^2}{\partial q_m \partial q_n} \left(\frac{\partial A_{mi}}{\partial M^{jn}} \right). \quad (3.18)$$

In term of the matrix M^{ij} it takes the form

$$D_{ij} = \frac{1}{84\lambda} \epsilon_{imm_1 m_2} \epsilon_{jnn_1 j_2} \frac{\partial^2}{\partial q_m \partial q_n} (M^{j_1 m_1} M^{j_2 m_2}). \quad (3.19)$$

And finally we derive an expression the generalized photon propagator via the constitutive tensor

$$D_{ij} = \frac{1}{84\lambda} \epsilon_{imm_1 m_2} \epsilon_{jnn_1 j_2} \left[\chi^{j_1(mn)m_1} \chi^{j_2(ab)m_2} + 2\chi^{j_1(ma)m_1} \chi^{j_2(nb)m_2} \right] q_a q_b. \quad (3.20)$$

For the Maxwell constitutive tensor, the second adjoint takes the form

$$B_{ijkl} = 2\lambda_0^2 g q^2 \left[(g_{ij} q_l q_k + g_{kl} q_i q_j) - (g_{il} q_j q_k + g_{kj} q_i q_l) \right]. \quad (3.21)$$

Calculating with (3.18) we come to the standard Feynman propagator expression.

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